Proof Without Words: The Pythagorean Theorem with Equilateral Triangles

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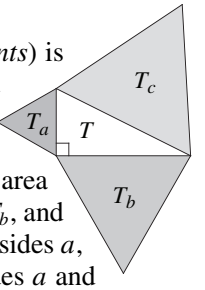


The Pythagorean theorem (Proposition I.47 in Euclid's *Elements*) is usually illustrated with squares drawn on the sides of a right triangle. However, as a consequence of Proposition VI.31 in the *Elements*, any set of three similar figures may be used, such as equilateral triangles as shown at the right. Let T denote the area of a right triangle with legs a and b, and hypotenuse c; let T_a, T_b, and T_c denote the areas of equilateral triangles drawn externally on sides a, b, and c; and let P denote the area of a parallelogram with sides a and b and 30° and 150° angles. Then we have

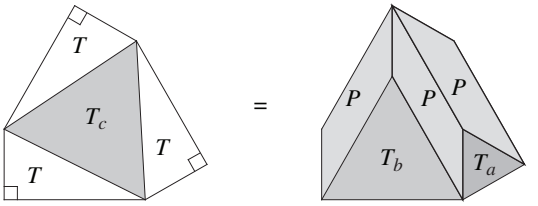
Lemma. $T = P$.

Proof.

$$T_a + 2T = T_a + 2P \Longrightarrow T = P.$$

Theorem. $T_c = T_a + T_b$.

Proof.



$$T_c + 3T = T_a + T_b + 3P \Longrightarrow T_c = T_a + T_b.$$

Summary. A visual proof of a modified Pythagorean theorem, showing that the area of an equilateral triangle constructed on the hypotenuse of a right triangle equals the sum of the areas of equilateral triangles constructed on the legs.

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